Perron-Frobenius-Ruelle Theorem

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A new metric on $X = X_A^b$: $d(x,y) = 2^{-\inf\{n \geq 0 : x_n \neq y_n\}} = 2^{-\text{max generation of a } b\text{-adic interval containing both } x \text{ and } y}$.

ultrametric: $d(x,z) \leq \max(d(x,y), d(y,z))$. Will consider only aperiodic A.

Let $f \in C(X)$. Define
$\omega_n(f) := \sup\{|f(x) - f(y)|,\ d(x,y) \leq 2^{-n-1}\}$ (i.e. maximal fluctuation on a n-th generation cylinder set!)
$f \in C(X) \Leftrightarrow \omega_n(f) \to 0$.

$B_0(X) := \{\|f\|_{B_0} := \sum_{n=0}^{\infty} \omega_n(f) + \|f\|_\infty < \infty\}$ - a Banach space.

$C_\delta \subset B_0$, $C_\delta := \{f \in C(X) : \exists C > 0\ \forall x, y \in X : |f(x) - f(y)| \leq C\, d(x,y)^\delta\}$
δ-Hölder functions $\Leftrightarrow \omega_n(f) \leq C\, 2^{-n\delta}$.

For $\varphi \in C(X)$, define Ruelle (transfer) operator on $C(X)$ by
$$L_\varphi(f)(x) := \sum_{y : Ty = x} e^{\varphi(y)} f(y) = \sum_{j=0}^{b} e^{\varphi(jx)} f(jx).$$

Observe: $L_\varphi^n(f)(x) = \sum_{y : T^n y = x} e^{S_n \varphi(y)} f(y)$.

Thm (Perron-Frobenius-Ruelle). Let $\varphi \in B_0(X)$. Then
1) $\exists$ simple eigenvalue $\beta > 0$ of L_φ and an associated eigenvector $h > 0$, $h \in C(X)$.
2) $\exists!$ probability measure μ on X: $L^*_\varphi(\mu) = \beta\mu$; $\forall v \in C(X)$, $\beta^{-n} L^n_\varphi(v)$ converges uniformly on X to $C_v h$, where $C_v = \frac{\int v\, d\mu}{\int h\, d\mu}$.
3) $\log \beta = P(\varphi)$ - pressure of φ.

We use:
Thm (Schauder-Tychonov). Let K be a convex compact subset of a locally convex topological vector space, $\Phi : K \circlearrowleft$ - continuous function. Then $\exists$ a fixed pt. of Φ: $x \in K$, $\Phi(x) = x$.

Pf (of PRF).
Step 1. $\exists \mu : L^*_\varphi \mu = \beta\mu$.
Let $C^*(X)$ be the space of signed measures, $M(X) \subset C^*(X)$ - a set of probability measures. It is compact (as a closed subset of closed unit ball) in weak* topology and convex.
Now let
$\beta(\mu) := \int L_\varphi(1)\, d\mu = \int \Big(\sum_{Ty = x} e^{\varphi(y)}\Big) d\mu(x) \geq e^{-\|\varphi\|_{B_0}}$ ($\|\cdot\|_\infty$ would be enough).
& $\Phi : \mu \to \frac{L^*_\varphi(\mu)}{\int L_\varphi(1)\, d\mu}$, $\Phi : M(X) \to M(X)$, weak*-continuous,
so Φ has a fixed pt. μ with $\boxed{L^*_\varphi(\mu) = \beta\mu}$. Fix μ! Let $\beta = \beta(\mu)$ for this μ.

Step 2: $\exists h > 0 : L_\varphi h = \beta h$

Let $x, y \in X$. Define
$$C(x,y) = \sup_{k \geq 1} \sup_{(z_1 \dots z_k) \text{ admissible}} \big(S_k \varphi(z_1 \dots z_k x) - S_k \varphi(z_1 z_2 \dots z_k y)\big),\quad C(x,y) \leq \|\varphi\|_{B_0}$$
maximal difference between S_k along the same preimages. $d(x,y) = 1$ (no common elements).
Observe that $\varphi(x) - \varphi(y)$ does not participate here!

Notice, that for any x, y,
$C(x,y) \leq \sum_{k}^{\infty} \omega_k(\varphi) \leq \|\varphi\|_{B_0}$
Also, if $d(x,y) \leq 2^{-n-1}$, then the beginnings of x and y are the same, so it can be improved to $C(x,y) \leq \sum_{k=n+1}^{\infty} \omega_k(\varphi)$
Observe also that for any $j \in \{0, \dots, b-1\}$,
$\varphi(jx) - \varphi(jy) \leq C(x,y) - C(jx, jy)$ (because to get from $C(jx, jy)$ to $C(x,y)$, you add $\varphi(jx) - \varphi(jy)$).
Observe also that if $\varphi \in C_\delta$, then $C(x,y) \leq C(d(x,y))^\delta$ (since the tails of eq. series decay exponentially).
Consider now
$$\Lambda = \Big\{g \in C(X) : g \geq 0,\ \int g\, d\mu = 1,\ \frac{g(x)}{g(y)} \leq e^{C(x,y)}\ \forall x, y\Big\}.$$
Notice that $\frac{\max g}{\min g} \leq e^{\|\varphi\|_{B_0}}$ for $g \in \Lambda$, so
$e^{-\|\varphi\|_{B_0}} \leq g \leq e^{\|\varphi\|_{B_0}}$, because $\int g\, d\mu = 1$.

Also $|g(x)-g(y)| \le \left(e^{\max(C(x,y),C(y,x))}-1\right)\|g\|_\infty$

Thus Λ is uniformly bounded, equicontinuous, closed, so, by Arzela-Ascoli, it is compact. It is also obviously convex.

4 $\tilde{L}: C(X) \to C(X)$ defined by $\tilde{L} := \frac{L_\varphi(g)}{\beta}$.

Let us now prove that $\tilde{L}(\Lambda) \subset \Lambda$. Indeed, for $g \in \Lambda$,

$$\int \tilde{L}(g)\,d\mu = \frac{1}{\beta}\langle L_\varphi(g), \mu\rangle = \langle g, \frac{L_\varphi^* \mu}{\beta}\rangle = \langle g, \mu\rangle = 1.$$

Also, $$L_\varphi(g)(x) = \sum_{j=0}^{b-1} e^{\varphi(jx)} g(jx) \le \sum_{j=0}^{b-1} e^{\varphi(jy)} g(jy) e^{\varphi(jx)-\varphi(jy)+C(jx,jy)} \le L_\varphi(g)(y)e^{C(x,y)}.$$

Thus $\tilde{L}(g) \in \Lambda$.

By Schauder-Tichonoff again, $\exists h \in \Lambda: \tilde{L}(h) = h$, i.e. $L_\varphi h = \beta h$. $h \in \Lambda$, so $h > 0$. In addition, if $\varphi \in C^\delta$, then $\log h \in C^\delta$.

Step 3: β is a simple eigenvalue of L_φ.

Let h_1 be another eigenvector for β, $h_1 \not\equiv 0$.
Then, by possibly replacing with $-h_1$, can assume that for some X, $h_1(x) > 0$.
Consider $\lambda_0 = \sup\{\lambda \ge 0: h(y) - \lambda h_1(y) \ge 0 \ \forall y \in X\} < \infty$, since $\lambda_0 \le \frac{h_0(x)}{h_1(x)}$. Then $h(y) - \lambda_0 h_1(y) \ge 0$, and for some z, $h(z) = \lambda_0 h_1(z)$.

$h_2(y) := h_1(y) - \lambda_0 h_1(y)$ is also β-eigenvector.
Then $0 = \beta h_2(z) = (L_\varphi h_2)(z) = \sum e^{\varphi(y)} h_2(y)$. Since $h_2(y) \ge 0$, $e^{\varphi(y)} > 0$, this implies that $h_2(y) = 0 \quad \forall y \in T^{-1}(z)$, $T(y) = z$
and, repeating the argument, for $y \in T^{-n}(z)$, which is dense in X (here we use aperiodicity). So $h_2 = h - \lambda_0 h_1 \equiv 0$.

Step 4: Normalizing φ.

Define $\tilde{\varphi} := \varphi - \log h \circ T + \log h - \log \beta$.
Define also $M_h f := hf$ - multiplication operator.
Then
$$L_{\tilde{\varphi}}(f)(x) = \sum_{T(y)=x} e^{\varphi(y)} \frac{1}{h(x)} \cdot h(y) \frac{1}{\beta} f(y) = \frac{1}{\beta} M_h^{-1} L_\varphi M_h f -$$
operator, conjugate to L_φ. $L_{\tilde{\varphi}}(1) = 1$, 1 is a simple eigenvalue. Since $L_{\tilde{\varphi}}$ is a positive operator (i.e. $g \ge 0 \Rightarrow L_{\tilde{\varphi}}(g) \ge 0$), this implies that $\|L_{\tilde{\varphi}}\| = 1$.
Also, if $\varphi \in C^\delta$, then $\tilde{\varphi} \in C^\delta$.
In general, we might have $\tilde{\varphi} \notin B_0$. Nevertheless, if $d(x,y) \le 2^{-n-1}$, then

$$\tilde{C}(x,y) := \sup_{k \ge 1} \sup_{(z_1,\dots,z_k)\text{-admissible}} \left(S_k\tilde{\varphi}(z_1\dots z_k x) - S_k\tilde{\varphi}(z_1 z_2 \dots z_k y)\right) =$$
$$\sup_{k \ge 1} \sup_{(z_1\dots z_k)\text{-admissible}} \big(S_k \varphi(z_1 \dots z_k x) - S_k \varphi(z_1 \dots z_k y) - \log h(x) + \log h(y) + \log h(z_1 \dots z_k x) - \log h(z_1 \dots z_k y)\big) \le$$
$$C(x,y) + 2\omega_n(\log h).$$

Thus, for $v \in C(X)$, $d(x,y) \le 2^{-k-1}$, we have
$$\left|L^n_{\tilde{\varphi}}(v)(x) - L^n_{\tilde{\varphi}}(v)(y)\right| \le \underbrace{\Big|\sum_{(z_1\dots z_n)\text{-admissible}} e^{S_n\tilde{\varphi}(z_1\dots z_n x)}\big(v(z_1 \dots z_n x) - v(z_1 \dots z_n y)\big)\Big|}_{\mathrm{I}} +$$
$$\underbrace{\Big|\sum v(z_1 \dots z_n y)\left(e^{S_n\tilde{\varphi}(z_1 \dots z_n x)} - e^{S_n\tilde{\varphi}(z_1 \dots z_n y)}\right)\Big|}_{\mathrm{II}}.$$

Note that $\mathrm{I} \le \omega_{k+n}(v) \underbrace{\sum e^{S_n\tilde{\varphi}}} = \omega_{k+n}(v)$

$$\mathrm{II} \le \|v\|_\infty \Big(\underbrace{\sum e^{S_n\tilde{\varphi}(z_1 \dots z_n y)}}_{=1}\Big)\left|e^{\tilde{C}(x,y)} - 1\right| = \|v\|_\infty \left|e^{\tilde{C}(x,y)} - 1\right|$$

Thus the family $(L^n_{\tilde{\varphi}} v)_n$ is equicontinuous.
Also, since $\|L_{\tilde{\varphi}}\| = 1$, it is uniformly bounded.
Assume that (n_k) is such that $L^{n_k}_{\tilde{\varphi}}(v)$ converges to some v^*. Then, since $L_{\tilde{\varphi}}$ is positive and $\|L_{\tilde{\varphi}}\| = 1$ we have:

$n \geq 0$, hence $\|L^n_{\tilde\varphi}\| = 1$, i.e. is uniformly bounded.

Assume that (n_k) is such that $L^{n_k}_{\tilde\varphi}(v)$ converges to some v^*. Then, since $L_{\tilde\varphi}$ is positive and $\|L_{\tilde\varphi}\| = 1$, we have: the sequence $\sup_x (L^n_{\tilde\varphi} v(x))$ is non-increasing, converges to $\sup v^* =: s$.

But for any N, $L^N_{\tilde\varphi}(v^*) = \lim_{k\to\infty} L^{N+n_k}_{\tilde\varphi}(v)$, so $\sup_x (L^N_{\tilde\varphi}(v^*)) = \inf_n \left(\sup_x \left(L^n_{\tilde\varphi} v\right)\right) = \sup_x v^*$, and $s \geq L^N_{\tilde\varphi} v^*$.

Thus, if $s = v^*(x)$ for some x, then $v^*(y) = s$ $\forall y : T^N(y) = x$.

By density of $(T^{-n}(x))$, $v^*(x) \equiv s$. Thus any convergent subsequence of precompact (by Arzela-Ascoli) $(L^n_{\tilde\varphi} v)$ converges to s. So $L^n_{\tilde\varphi}(v) \to s(v)$ uniformly.

Note also, that $s(v)$ is linear, $|s(v)| \leq \|s\|_\infty$. (decompose $s = s_+ - s_-$). So $s(v) = \int v \, d\nu$ for some probability measure ν. [$s(1) = 1$] Also, $s(L_{\tilde\varphi} v) = s(v)$ (since $\lim L^{n+1} = \lim L^n_{\tilde\varphi}$). Thus $L^*_{\tilde\varphi} \nu = \nu$. If ν_1 is another measure with $L^*_{\tilde\varphi} \nu_1 = \nu_1$, then $\forall v \in C(X)$, $\int L^n_{\tilde\varphi} v \, d\nu_1 = \int v \, d\nu_1$. Taking $n \to \infty$, we get that $\nu_1(X) \int v \, d\nu = \int \lim_{n\to\infty} L^n_{\tilde\varphi} v \, d\nu_1 = \int v \, d\nu_1$, thus $\nu_1 = \nu_1(X)\nu$.

Step 5: $\exists!\, \mu : L^*_\varphi \mu = \beta\mu$. Also $\forall v \in C(X)$,
$$\beta^{-n} L^n_\varphi(v) \to h \frac{\int v \, d\mu}{\int h \, d\mu} \text{ uniformly.}$$

Notice that $L^*_\varphi \mu = \beta\mu \Leftrightarrow L^*_{\tilde\varphi}(h\mu) = h\mu$.

(since $L_{\tilde\varphi} = \frac{1}{\beta} M_h^{-1} L_\varphi M_h$). Thus $h\mu = \nu$.

Also $\underbrace{\beta^{-n} L^n_\varphi(v)}_{M_h L^n_{\tilde\varphi} M_h^{-1} v} \to M_h \int \frac{v}{h} d\nu = h \int v \, d\mu$.

Now, let us remember that our h is normalized by $\int h \, d\mu = 1$.

Step 6: $P(\varphi) = \log \beta$.

$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{C \in A_n} e^{(S_n\varphi)_C} \right).$$

If $x \in X$, then
$$|(S_n\varphi)_C - S_n\varphi(cx)| \leq \sum_{k=1}^n \omega_k(\varphi) \leq \|\varphi\|_{B_0}.$$

Thus $P(\varphi) \leq \lim_{n\to\infty} \left(\frac{1}{n} \log \sum_{T^n y} e^{S_n\varphi(y)} + \frac{1}{n}\|\varphi\|_{B_0} \right) = \lim_{n\to\infty} \frac{1}{n} \log L^n_\varphi(1)(x) = \log\beta$

But $P(\varphi)$ is clearly $\geq$, since $(S_n\varphi)_C \geq S_n\varphi(y)$ $\forall y \in C$.

When $\varphi \in C^\delta$, we can say more. Note that in this case h and $\tilde\varphi \in C^\delta$.

Also, $L_\varphi(C^\delta) \subset C^\delta$. Equip C^δ with norm $\|v\|_\delta = \|v\|_\infty + \sup 2^{n\delta}\omega_n(v)$.

$\tilde\varphi := \varphi - \log h \circ T + \log h - \log\beta$.

Thm. $\varphi \in C^\delta$. $\exists\, r < \beta$: $sp(L_\varphi|_{C^\delta}) \setminus \{\beta\} \subset B(0,r)$.

Pf. $L_\varphi = \beta M_h^{-1} L_{\tilde\varphi} M_h$. So enough to establish it for $\tilde\varphi$ with $L_{\tilde\varphi}(1) = 1$.

$\nu : L^*_{\tilde\varphi} \nu = \nu$. Write uniquely every $v \in C^\delta$ as $v = \lambda + \mu v_2$, where

$v_2 \in V := \{ v \in C^\delta, \int v\, d\nu = 0, \|v\|_\delta \le 1 \}$

Let us investigate $sp(L_{\tilde\varphi}|_{\mu\nu}) = \inf \{ \|(L^\infty_{\tilde\varphi}|_{\mu\nu})^N\|^{1/N}, N \ge 1 \}$.

Observe that V is a compact subset of $C(X)$, by Arzela-Ascoli.

Also, for $v \in V$, $L^n_{\tilde\varphi}(v) \to \int v\, d\nu = 0$ uniformly. Thus $\sup_V \|L^n \varphi\|_\infty \to 0$.

Also, if $v \in V$, then

$\forall N, n \ge 1, k \ge 0: \omega_k(L^{n+N}_{\tilde\varphi}(v)) \le \underbrace{\omega_{k+n}(L^N_{\tilde\varphi}(v))}_{I} + \underbrace{2\|L^N_{\tilde\varphi}(v)\|_\infty}_{II} \underbrace{\omega_k(\tilde\varphi)}_{\le 2^{-k\delta}}$.

$II \le \frac{1}{4} 2^{-k\delta}$ for large N.

Now, $I \le 6^N 2^{-(n+k)\delta} \le 6^N 2^{-n\delta} 2^{-k\delta} \le \frac{1}{4} 2^{-k\delta}$ for large n.

Thus $\|L^{n+N}_{\tilde\varphi}(v)\|_\delta \le \frac{1}{2} + \|L^{n+N}_{\tilde\varphi}(v)\|_\infty \le \frac{3}{4}\ \forall v \in V$.

So $\|L^{n+N}_{\tilde\varphi}|_{\mu\nu}\| \le \frac{3}{4}$, and $sp(L_{\tilde\varphi}|_{\mu\nu}) \le \left(\frac{3}{4}\right)^{1/(N+n)} < 1$ ∎

$sp\, A = \lim_{n\to\infty} \|A^n\|^{1/n} = \inf \|A^n\|^{1/n}$

So, for $\varphi \in C^\delta$, L_φ has an isolated maximal eigenvalue. This allows us to use the following theorem to establish nice dependence on parameters.

Thm (Kato). Let B be a Banach space, $L: B \to B$ - continuous operator such that $sp(L) = \{\lambda(L)\} \cup S$, where $S \subset B(0,r)$, $r < \lambda(L)$, $\lambda(L)$ be a simple eigenvalue with eigenvector $v(L)$.

Then

1) $\forall \varepsilon < \lambda(L) - r\ \exists \delta: \|M - L\| < \delta \Rightarrow sp(M) \cap B(\lambda(L), \varepsilon) = \lambda(M)$, where $\lambda(M)$ is a simple eigenvalue, $\lambda(M) = sp(M)$.

2) For any real-analytic $f: U \to B$, $U \subset \mathbb{R}^k$, $f(x_0) = L$, $\exists \delta > 0$: $B(x,\delta) \subset U$, for $y \in B(x,\delta)$ $y \to \lambda(f(y))$ and $y \to v(f(y))$ are real-analytic.

As an immediate application of Kato Thm, we have.

Thm Let $f: U \to C^\delta$ is real-analytic. Then $P(f(x))$, $\mu(f(x))$, $h(f(x))$, $\tilde\varphi(f(x))$, $\nu(f(x))$ are real-analitic in U.